P 177 - 178

1. Let $g(z) = e^{f(z)}$

Since f is entire, so is g.

Also $|g(z)| = |e^{f(z)}| = |e^{u(x,y) + iv(x,y)}|$

$$= e^{u(x,y)}$$

$$\leq e^{u_0}$$

By Liouville's Theorem, g is constant.

Hence, u is constant. $\square$

2. Let $g(z) = \frac{1}{f(z)}$.

Since f is analytic and $f(z) \neq 0$ anywhere in R, g is well-defined and analytic in R.

By Maximum Principle, $|g(z)| = \frac{1}{|f(z)|}$ has a maximum value in R which occurs on the boundary of R and never in the interior.

Hence, $|f(z)|$ has a minimum value in R which occurs on the boundary of R and never in the interior. $\square$

3. Let R be a closed bounded region with 0 in the interior of R and $f(z) = z$

Clearly, f is analytic on R.

But $\min\limits_{z \in R} |f(z)| = |f(0)| = 0$ and 0 is an interior point of R.

$\square$

4. Let $f(z) = \sin z$, then $|f(z)|^2 = \sin^2 x + \sinh^2 y$.

$|f(z)|$ attains its maximum

$\Leftrightarrow |f(z)|^2$ attains its maximum

$\Leftrightarrow$ $\sin^2 x$ and $\sinh^2 y$ attains their maximum

$\Leftrightarrow$ $z = \frac{\pi}{2} + i$.

$\square$

5. Let $g(z) = e^{f(z)}$.

Since f is analytic on R, g is analytic and $g(z) \neq 0$ anywhere on R.

By (3), $|g(z)| = e^{u(x,y)}$ has a minimum value in

R which occurs on the boundary of R and never in the interior.

So does $u(x,y)$.

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2. (a) $e^z = \sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!}(z-1)^n$

$$= \sum_{n=0}^{\infty} \frac{e}{n!}(z-1)^n$$

$$= e\sum_{n=0}^{\infty} \frac{(z-1)^n}{n!}$$

(b) $e^z = e \cdot e^{z-1}$

$$= e \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}(z-1)^n$$

$$= e \sum_{n=0}^{\infty} \frac{(z-1)^n}{n!}$$

$\square$

4. $\cos z = -\sin(z - \frac{\pi}{2})$

$$= -\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}(z-\frac{\pi}{2})^{2n+1}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n+1)!}(z-\frac{\pi}{2})^{2n+1}$$

$\square$

9. $f(z) = \sin z^2 = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}(z^2)^{2n+1}$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{4n+2}$$

$$= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n$$

By comparing the coefficients, $f^{(4n)}(0) = 0$ and

$f^{(2n+1)}(0) = 0, \ \forall n \in \mathbb{N}$. $\square$

10. (a) $\sinh z = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}$

$$\frac{\sinh z}{z^2} = \sum_{n=0}^{\infty} \frac{z^{2n-1}}{(2n+1)!} = \frac{1}{z} + \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+3)!}$$

(b) $\sin z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!}$

$$\frac{\sin z}{z^2} = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n-1}}{(2n+1)!}$$

$$\frac{\sin z^2}{z^4} = \sum_{n=0}^{\infty} \frac{(-1)^n z^{4n-2}}{(2n+1)!} = \frac{1}{z^2} - \frac{z^2}{3!} + \frac{z^6}{5!} - \frac{z^{10}}{7!} + \cdots$$

$\square$

11. $$\frac{1}{4z-z^2} = \frac{1}{4z}\left(\frac{1}{1-\frac{z}{4}}\right)$$

$$= \frac{1}{4z}\left(\sum_{n=0}^{\infty}\left(\frac{z}{4}\right)^n\right) \quad , \ \forall\ 0<|z|<4$$

$$= \frac{1}{4z} + \sum_{n=0}^{\infty}\frac{z^n}{4^{n+2}} \quad , \ \forall\ 0<|z|<4 .$$

$\square$